Ulf Söderström
IGIER, Università Bocconi

January 18, 2006

A simple model for monetary policy analysis

This note derives a simple model that we will use to analyze monetary policy. In the complete model a very large number of firms act under monopolistic competition, each producing a slightly different variety of goods, so that each firm has some pricing power. These firms set prices optimally by maximizing profits, but prices are sticky and cannot be changed in every period. Households are identical, live forever and own the firms, and choose consumption optimally in order to maximize life-time utility.

In the simple version of the model used here, we will disregard households' optimal consumption choice across goods as well as their choice between labor and leisure. We will also simplify firms' optimization problem. For a complete derivation of the model, see Walsh (2003, Ch. 5.4).

Households

Households choose consumption C_t to maximize their discounted sum of expected life-time utility:

$$\mathrm{E}_t \sum_{k=0}^{\infty} \beta^{t+k} u(C_{t+k}), \tag{1}$$

where β is the discount factor, which is close to but below 1. The utility function depends only on consumption according to

$$u(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}, \tag{2}$$

where $\sigma > 0$ is the inverse of the elasticity of intertemporal substitution.[1] Note that the utility function is concave: it is increasing in C_t but at a decreasing rate, so the marginal utility of consumption $u'(C_t) = C_t^{-\sigma}$ is decreasing in C_t.

Households use their income (profits from firms) to either consume goods or save in one-period nominal bonds, denoted B_t, which pay the nominal interest rate i_t. Thus the budget constraint (in real terms) is given by

$$C_t + \frac{B_t}{P_t} = \frac{(1+i_{t-1})B_{t-1}}{P_t} + \Pi_t, \tag{3}$$

[1] This elasticity is given by

$$\epsilon \equiv -\frac{u'(C_t)}{u''(C_t)C_t} = -\frac{C_t^{-\sigma}}{-\sigma C_t^{-(1+\sigma)}C_t} = \frac{1}{\sigma}.$$

where P_t is the aggregate price level and Π_t are real profits from firms.

Using the utility function (2) we can rewrite the households' problem of maximizing life-time utility in equation (1) with respect to C_t and B_t subject to the budget constraint (3) as the Lagrangian

$$\mathrm{E}_t \sum_{k=0}^{\infty} \beta^{t+k} \left\{ \frac{C_{t+k}^{1-\sigma}}{1-\sigma} - \lambda_{t+k} \left[C_{t+k} + \frac{B_{t+k}}{P_{t+k}} - (1+i_{t+k-1}) \frac{B_{t+k-1}}{P_{t+k}} - \Pi_{t+k} \right] \right\}, \quad (4)$$

where λ_t is the multiplier on the budget constraint at t. Writing out the terms that involve C_t and B_t gives

$$\begin{aligned} \mathrm{E}_t \Bigg\{ & \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \lambda_t \left(C_t + \frac{B_t}{P_t} - (1+i_{t-1}) \frac{B_{t-1}}{P_t} - \Pi_t \right) \right] \\ + \; & \beta^{t+1} \left[\frac{C_{t+1}^{1-\sigma}}{1-\sigma} - \lambda_{t+1} \left(C_{t+1} + \frac{B_{t+1}}{P_{t+1}} - (1+i_t) \frac{B_t}{P_{t+1}} - \Pi_{t+1} \right) \right] + \dots \Bigg\}, \end{aligned} \quad (5)$$

so the first-order conditions with respect to C_t and B_t are given by

$$C_t \; : \; \beta^t C_t^{-\sigma} - \beta^t \lambda_t = 0, \quad (6)$$

$$B_t \; : \; -\beta^t \lambda_t \frac{1}{P_t} + \beta^{t+1} \mathrm{E}_t \left[\lambda_{t+1} (1+i_t) \frac{1}{P_{t+1}} \right] = 0. \quad (7)$$

The first-order condition for C_t implies that the multiplier λ_t satisfies

$$\lambda_t = C_t^{-\sigma}, \quad (8)$$

so the multiplier is equal to the marginal utility of consumption. Using this to eliminate λ_t and $\mathrm{E}_t \lambda_{t+1}$ in the first-order condition for B_t gives

$$-C_t^{-\sigma} \frac{1}{P_t} + \beta \mathrm{E}_t \left[C_{t+1}^{-\sigma} (1+i_t) \frac{1}{P_{t+1}} \right] = 0, \quad (9)$$

which can be rearranged to give the *consumption Euler equation*

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left[(1+i_t) \frac{P_t}{P_{t+1}} C_{t+1}^{-\sigma} \right]. \quad (10)$$

The Euler equation can also be written in terms of the real interest rate r_t as

$$C_t^{-\sigma} = \beta \mathrm{E}_t \left[(1+r_t) C_{t+1}^{-\sigma} \right], \quad (11)$$

where

$$\mathrm{E}_t (1+r_t) \equiv \mathrm{E}_t \left[(1+i_t) \frac{P_t}{P_{t+1}} \right] = \mathrm{E}_t \left[\frac{1+i_t}{1+\pi_{t+1}} \right], \quad (12)$$

and π_t is the rate of inflation, defined as

$$\pi_t \equiv \frac{P_t - P_{t-1}}{P_{t-1}}. \tag{13}$$

Equation (11) tells us how consumption is allocated optimally over time depending on the real interest rate r_t. An increase in r_t makes it more attractive to save in bonds, so the household reduces current consumption C_t relative to expected future consumption $\mathrm{E}_t C_{t+1}$.[2]

The consumption Euler equation in (10) or (11) is a non-linear expression, which is very cumbersome to work with. Therefore, one usually rewrites the model in terms of a log-linear expression, which is linear in the natural logarithms of the variables.[3] We will use lower-case letters to denote logs, so $c_t \equiv \log C_t$.

Then we can write equation (11) as

$$-\sigma c_t = \log \beta + \log \mathrm{E}_t \left[\frac{1 + i_t}{1 + \pi_{t+1}} \right] - \sigma \mathrm{E}_t c_{t+1}. \tag{14}$$

Since β is close to 1, $\log \beta$ is close to zero, and if i_t and π_{t+1} are fairly small, we can use the approximations

$$\begin{aligned} \log \mathrm{E}_t \left[\frac{1 + i_t}{1 + \pi_{t+1}} \right] &\approx \log \mathrm{E}_t \left[1 + i_t - \pi_{t+1} \right] \\ &\approx \log \left[1 + i_t - \mathrm{E}_t \pi_{t+1} \right] \\ &\approx i_t - \mathrm{E}_t \pi_{t+1}. \end{aligned} \tag{15}$$

Then we can approximate the consumption Euler equation as

$$-\sigma c_t = -\sigma \mathrm{E}_t c_{t+1} + \left[i_t - \mathrm{E}_t \pi_{t+1} \right], \tag{16}$$

or

$$c_t = \mathrm{E}_t c_{t+1} - \frac{1}{\sigma} \left[i_t - \mathrm{E}_t \pi_{t+1} \right]. \tag{17}$$

Finally, we will write the Euler equation in terms of the output gap x_t, which is the percent deviation of output from its natural level:[4]

$$x_t = y_t - y_t^n, \tag{18}$$

[2] If r_t increases, the right-hand side of (11) increases, so also the left-hand side must increase. Since $\sigma > 0$, $C_t^{-\sigma}$ increases if C_t falls.

[3] In practice, non-linear models are typically log-linearized around their steady state, so all variables are interpreted as log deviations from their steady-state levels.

[4] For small values, percent differences are approximately equal to log differences.

where y_t^n is the natural level of output. First we use the national accounting identity $y = c + g$, where g is government spending[5] to write the consumption Euler equation in terms of the log level of output, y_t, as

$$y_t - g_t = \mathrm{E}_t y_{t+1} - \mathrm{E}_t g_{t+1} - \frac{1}{\sigma}\left[i_t - \mathrm{E}_t \pi_{t+1}\right]. \tag{19}$$

Then use the definition of the output gap to obtain

$$x_t + y_t^n - g_t = \mathrm{E}_t x_{t+1} + \mathrm{E}_t y_{t+1}^n - \mathrm{E}_t g_{t+1} - \frac{1}{\sigma}\left[i_t - \mathrm{E}_t \pi_{t+1}\right]. \tag{20}$$

This gives us the final expression

$$x_t = \mathrm{E}_t x_{t+1} - \frac{1}{\sigma}\left[i_t - \mathrm{E}_t \pi_{t+1}\right] + u_t, \tag{21}$$

where u_t is a "demand shock" defined as

$$u_t \equiv (g_t - \mathrm{E}_t g_{t+1}) - \left(y_t^n - \mathrm{E}_t y_{t+1}^n\right). \tag{22}$$

Equation (21) is similar to a textbook IS or AD equation: it relates output to the real interest rate. But in contrast to the standard textbook model, the real interest rate affects households' decision to allocate consumption over time, not firms' investment decision. In addition, our AD equation is written in terms of the output gap, not the level of output. We will use this equation as our model of aggregate demand.

Firms

Firms act under imperfect competition, so they do not have to set their price equal to marginal cost, but are free to set their price optimally. It has been noted that firms in practice do not change their price continuously as demand conditions change, but seem to change prices only occasionally, for example once or twice a year. We formalize such "sticky prices" by assuming that a given firm faces a constant probability $(1-\omega)$ of being able to change its price in each period, as in Calvo (1983). As there are many firms in the economy, this implies that a fraction $(1-\omega)$ of firms are able to adjust its price in any period, while a fraction ω keeps its price fixed. The expected time between price adjustments the is $1/(1-\omega)$.[6]

Assume that if prices were flexible firm i would ideally like to set the target price p_t^*. When prices are sticky it cannot achieve this, but it wants to keep its actual

[5] Note that we have a closed economy with no capital accumulation, so imports, exports and investment are all zero.

[6] Firms' opportunities to adjust follow a Poisson process.

price p_{it} as close as possible to the target price.[7] Now consider a firm that is allowed to adjust its price at time t. This firm will take into account that it may not be able to change its price again for some time: the probability that the price chosen at t will still be valid after j periods is ω^j. Assuming that the cost of deviating from the target price is quadratic, the firm will set its price p_{it} to minimize

$$\sum_{j=0}^{\infty} \omega^j \beta^j \mathrm{E}_t \left(p_{it} - p^*_{t+j}\right)^2, \tag{23}$$

where β is firms' discount factor (the same as for households).

Denoting the optimal price by $\hat{p}_t$, the first-order condition is[8]

$$2\sum_{j=0}^{\infty} \omega^j \beta^j \mathrm{E}_t \left(\hat{p}_t - p^*_{t+j}\right) = 0. \tag{24}$$

Moving $\hat{p}_t$ outside the summation we obtain

$$\hat{p}_t \sum_{j=0}^{\infty} \omega^j \beta^j = \sum_{j=0}^{\infty} \omega^j \beta^j \mathrm{E}_t p^*_{t+j}, \tag{25}$$

and using the fact that

$$\sum_{j=0}^{\infty} \omega^j \beta^j = \frac{1}{1-\omega\beta} \tag{26}$$

as both ω and β are between zero and one, the optimal price is given by

$$\hat{p}_t = (1-\omega\beta) \sum_{j=0}^{\infty} \omega^j \beta^j \mathrm{E}_t p^*_{t+j}. \tag{27}$$

When the firm takes into account that it may not be able to change its price again for some time, its optimal price is a weighted average of current and expected future target prices. The larger is the probability that the firm will be able to reset its price soon (the smaller is ω), the smaller is the weight on the expected future target price.

Noting that

$$\mathrm{E}_t \hat{p}_{t+1} = (1-\omega\beta) \sum_{j=0}^{\infty} \omega^j \beta^j \mathrm{E}_t p^*_{t+j+1}, \tag{28}$$

we can break up the sum in equation (27) and write

$$\begin{aligned} \hat{p}_t &= (1-\omega\beta) p^*_t + (1-\omega\beta) \sum_{j=0}^{\infty} \omega^{j+1} \beta^{j+1} \mathrm{E}_t p^*_{t+j+1} \\ &= (1-\omega\beta) p^*_t + \omega\beta \mathrm{E}_t \hat{p}_{t+1}. \end{aligned} \tag{29}$$

[7] All variables are expressed in natural logarithms. As all firms are identical, their target price is the same, so p^*_t has no subscript i.

[8] Again the optimal price is the same for all firms that adjust at t, so $\hat{p}_t$ has no subscript i.

Now assume that the target price p_t^* depends on the aggregate price level p_t, the aggregate level of output y_t, and a supply shock ε_t according to

$$p_t^* = p_t + \gamma y_t + \varepsilon_t, \tag{30}$$

where $\gamma > 0$. Then

$$\hat{p}_t = (1-\omega\beta)\left[p_t + \gamma y_t + \varepsilon_t\right] + \omega\beta \mathrm{E}_t \hat{p}_{t+1}. \tag{31}$$

As a fraction $(1-\omega)$ of firms sets the optimal price $\hat{p}_t$ while a fraction ω keeps its price fixed, the aggregate price level follows

$$p_t = (1-\omega)\hat{p}_t + \omega p_{t-1}, \tag{32}$$

which we can solve for the optimal price $\hat{p}_t$ to obtain

$$\hat{p}_t = \frac{1}{1-\omega} p_t - \frac{\omega}{1-\omega} p_{t-1} \tag{33}$$

Using this in the expression for $\hat{p}_t$ and $\mathrm{E}_t\hat{p}_{t+1}$ in equation (31) gives

$$\begin{aligned} &\frac{1}{1-\omega} p_t - \frac{\omega}{1-\omega} p_{t-1} \\ &= (1-\omega\beta)\left[p_t + \gamma y_t + \varepsilon_t\right] + \omega\beta\left[\frac{1}{1-\omega}\mathrm{E}_t p_{t+1} - \frac{\omega}{1-\omega} p_t\right] \\ &= (1-\omega\beta)\left[\gamma y_t + \varepsilon_t\right] + \frac{\omega\beta}{1-\omega}\mathrm{E}_t p_{t+1} + \frac{1-\omega-\omega\beta}{1-\omega} p_t \\ &= (1-\omega\beta)\left[\gamma y_t + \varepsilon_t\right] + \frac{\omega\beta}{1-\omega}\mathrm{E}_t \pi_{t+1} + p_t, \end{aligned} \tag{34}$$

where

$$\pi_t \equiv p_t - p_{t-1} \tag{35}$$

is the rate of inflation between periods $t-1$ and t.

Moving p_t to the left-hand side gives

$$\frac{\omega}{1-\omega}\pi_t = (1-\omega\beta)\left[\gamma y_t + \varepsilon_t\right] + \frac{\omega\beta}{1-\omega}\mathrm{E}_t \pi_{t+1}, \tag{36}$$

and solving for π_t gives

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \frac{(1-\omega\beta)(1-\omega)}{\omega}\left[\gamma y_t + \varepsilon_t\right]. \tag{37}$$

Finally, using the definition of the output gap x_t, we can write

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + e_t, \tag{38}$$

where

$$\kappa \equiv \frac{(1-\omega\beta)(1-\omega)\gamma}{\omega}, \tag{39}$$

and e_t is a "supply shock" defined as

$$e_t = \kappa y_t^n + \frac{(1-\omega\beta)(1-\omega)}{\omega}\varepsilon_t. \tag{40}$$

Equation (38) is called the "New-Keynesian Phillips curve." While traditional Phillips curves describe a negative relationship between the rate of inflation and the deviation of the rate of unemployment from the natural rate, the NKPC describes a positive relationship between inflation and the deviation of output from its natural level.[9] Of course, assuming an "Okun's law" relationship, that is, a negative relationship between the unemployment gap and the output gap, the two Phillips curves are consistent with each other.

Summary

The New-Keynesian model is summarized by two equations for inflation and the output gap:

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + e_t, \tag{41}$$

$$x_t = \mathrm{E}_t x_{t+1} - \frac{1}{\sigma}\left[i_t - \mathrm{E}_t \pi_{t+1}\right] + u_t. \tag{42}$$

This model is similar to the textbook AS-AD model, with a few important differences.

First, the New-Keynesian model is derived from a model with optimizing agents, so we know where the relationships come from, and the parameters and the shocks have a precise interpretation. For instance, the slope of the Phillips curve, κ, depends on the degree of price stickiness ω, the discount factor β, and the sensitivity of the target price to output, γ. More price stickiness implies that ω is larger, leading to a smaller κ, so inflation is less sensitive to the output gap.

Second, the optimizing behavior of rational firms and households implies that the current state of the economy (π_t, x_t) depends on expectations about the future $(\mathrm{E}_t \pi_{t+1}, \mathrm{E}_t x_{t+1})$. To see the implication of this, note that the Phillips curve implies

[9] We derived the New-Keynesian Phillips curve assuming a particular type of price stickiness following Calvo (1983). Roberts (1995) shows that two alternative models of sticky prices due to Taylor (1980) and Rotemberg (1982) give essentially the same expression for the Phillips curve.

that expected inflation is given by

$$\mathrm{E}_t\pi_{t+1} = \beta\mathrm{E}_t\pi_{t+2} + \kappa\mathrm{E}_t x_{t+1} + \mathrm{E}_t e_{t+1}, \tag{43}$$

and repeatedly substituting for expected inflation gives

$$\begin{aligned} \pi_t &= \kappa x_t + e_t + \beta\mathrm{E}_t\left[\kappa x_{t+1} + e_{t+1} + \beta\mathrm{E}_{t+1}\left[\kappa x_{t+2} + e_{t+2} + \beta\mathrm{E}_{t+3}\left[\ldots\right.\right.\right. \\ &= \sum_{j=0}^{\infty}\beta^j\mathrm{E}_t\left[\kappa x_{t+j} + e_{t+j}\right], \end{aligned} \tag{44}$$

so inflation at t depends on expectations of the output gap and the supply shock in the entire future. Likewise we can solve the output equation forward to obtain

$$x_t = \sum_{j=0}^{\infty}\beta^j\mathrm{E}_t\left\{-\frac{1}{\sigma}\left[i_{t+j} - \mathrm{E}_t\pi_{t+j+1}\right] + u_{t+j}\right\}, \tag{45}$$

so the output gap at t depends on expectations of the real short-term interest rate and the demand shock in the entire future. As we shall see, the importance of expectations has strong implications for the conduct of monetary policy and means that the ability of the central bank to affect private sector expectations is very important.

References

Calvo, Guillermo A. (1983), "Staggered prices in a utility-maximizing framework," *Journal of Monetary Economics*, 12 (3), 383–398.

Roberts, John M. (1995), "New Keynesian economics and the Phillips curve," *Journal of Money, Credit, and Banking*, 27 (4), 975–984.

Rotemberg, Julio J. (1982), "Monopolistic price adjustment and aggregate output," *Review of Economic Studies*, 49 (4), 517–531.

Taylor, John B. (1980), "Aggregate demand dynamics and staggered contracts," *Journal of Political Economy*, 88 (1), 1–23.

Walsh, Carl E. (2003), *Monetary Theory and Policy*, The MIT Press, second edition.